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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE V. RIGHT ANGLES AND ALLIED TOPICS.

Theorem 29. *The two portions of the surface into which the surface is divided by a line are superposable, that is, congruent: and may be called hemispheres.*

Theorem 30. *All hemispheres are congruent.*

Theorem 31. *The order of three directions a , b , c , issuing from a point O is the same as the order of the three corresponding opposite directions a' , b' , c' , issuing from that point.*

To show that two portions of surface coincide it is necessary to demonstrate that every point in one of them lies in the other, and conversely. Or, since this has presumably been done once in Theorem 27, it will merely be necessary to show that the boundaries of the portions of surface coincide, provided those boundaries are convex polygons or the other figures specified in the statement of Theorem 27. To demonstrate Theorem 29 let l be the given line, O any point of it, a and a' the two opposite directions which issue from O . Cause a and a' to rotate about the point O with the whole surface until a and a' fall respectively on the opposite directions a' and a . The line l has been moved into itself in such a way that what was originally on one side of the line has been moved to the other side. Theorem 27 will now apply. Theorem 30 is proved in a similar manner. The last of the above three theorems may also be stated by saying that if the direction b issues into the angle formed by a and c , the opposite direction issues into the vertical angle. Were the theorem untrue, the whole line formed of b and b' would lie in one hemisphere. This is impossible.

In the Second Lecture it was pointed out that a transformation which undid the change of position already brought about by some transformation, was called the inverse of that transformation. If the transformation undoes itself, that is to say, if the repetition of a transformation causes all points to take on their original positions the transformation is said to be *involutory*.

Theorem 32. *If an involutory transformation carries the geometric configuration α into the configuration β , it must conversely carry β into α .*

Theorem 33. *The transformation of rotating the surface about the point O until a given direction a comes to fall on its opposite direction a' is involutory.*

Theorem 34. *Vertical angles are congruent.*

The first of these theorems is easily seen to be merely another way of stating the definition of involutory transformation, the second is also obvious inasmuch as the repetition of the rotation brings the direction a into its original position so that Theorem 1 may be applied. To demonstrate the last theorem, let (a, b) be an angle and (a', b') the vertical angle. Rotate the direction a about the vertex of the angle until it comes to fall on the opposite direction a' . It must be proved that b falls on b' . Suppose that b falls on c . As the transformation is involutory c must be carried into b . The direction b' which is the continuation of b must fall on the direction c' which is the continuation of c (by Axiom V_2 , as interpreted under Theorem 2). Hence the directions a, b, c' go over respectively into the directions a', c, b' . Theorem 31 shows that the order of a, b, c' is the same as that a', b', c and consequently opposite to the order of a', c, b' , except in case c and b' coincide. As the motion of carrying a into a' cannot change the order of corresponding directions (Theorem 28) we are forced to conclude that c and b' coincide and the theorem is proved.

A careful comparison of the ideas underlying this demonstration and those on which the familiar proofs of this theorem depend is very instructive. It throws a great deal of light on how tempting it is to introduce a host of ideas which are in themselves really nothing short of new axioms. One very common proof is as follows: The angle (a, a') is a straight angle; (b, b') is likewise a straight angle; these angles contain the common angle (b, a') ; subtracting this common angle we have the desired relation, namely that the angle (a, b) is equal to (a', b') . This "proof" assumes that angles are possessed of magnitude which is a measurable quantity and that equal angles have the same measure and that these measures may be added and subtracted as ordinary numbers are added and subtracted. To prove these assumptions is no easy task. It is no simple matter to state how angles are to be measured. Exactly in what the idea of measurability consists is difficult to say. It is well to preserve as far as possible the distinction between pure demonstrational geometry and mensuration. Half a century ago there used to be placed in the back of books on arithmetic a long and important chapter on the mensuration of plane and solid figures. Here the student with very little worry over geometry learned to use those properties of figures with which every carpenter, plumber, grocer, or wide awake boy is of necessity more or less familiar. To do such problems the crude intuition and a few rules

sufficed. More recently there has seemed to be a tendency to make mensuration depend upon a previous training in demonstration and thus to put it off until the student shall have made the acquaintance with the fourth, fifth, and ninth books of geometry. This is logical and might be reasonable were it not for the fact that very frequently a training in mensuration does more for one in the way of pure geometry than the training in demonstration does for mensuration. The too great insistence on logical precision has an unfortunate effect of benumbing and repelling the young mind. For this reason the very recent movement on the part of teachers from the kindergartens up to the colleges toward a separation of the constructional parts of geometry and of mensuration from the demonstration-al part of the subject, allowing the latter to come later, cannot but be welcomed. From our point of view, which is that of strict logic, the possibility of constructing or measuring a figure is of secondary importance.

Another proof of the theorem that vertical angles are equal depends on the possibility of folding the angle over onto its vertical. Thus if b be folded over so as to coincide with a' , a which is the continuation of a' will fall along b' which is the continuation of b , and the angles will coincide throughout. Such a method of demonstration is entirely out of the question in spherical geometry where it is necessary carefully to distinguish between congruent and symmetrical angles and triangles. The fact that an angle (a, b) is not congruent to the angle (b, a) introduces a radical difference between the presentation of spherical geometry and the usual presentations of plane geometry. Strictly speaking the distinction between (a, b) and (b, a) should be preserved in plane geometry: for it is only by going out of the plane and assuming at least some of the geometry of space that an angle can be carried into itself in such a manner as to interchange the sides. No one who has not attempted to think through the first books of plane geometry with the distinction between symmetry and congruence in mind can have any realizing appreciation of how closely the ideas are interwoven and how important the relations of symmetry really are. Even in so careful a work as Hilbert's *Grundlagen der Geometrie* the ideas are nowhere well distinguished and the statement of one of the axioms (IV₄, page 11), where the possibility of interchanging the sides of an angle is placed in a parenthesis as if it were an unimportant alternative, is a good illustration of the lack of attention which the subject receives. It is to be noted that in the usual definition of a right angle as an angle which is equal to its supplement the word "equal" undoubtedly stands for "symmetric." This will be far clearer after the definition of right angle dependent solely on congruence and capable of use on the surface of the sphere has been given.

Theorem 35. *Given an angle (a, c) there exists one and only one direction b within the angle and such that the angle (a, b) is congruent to the angle (b, c) .*

This theorem shows that any angle may be bisected although no construction for the bisector is given. The proof of the theorem is not much different from that given for Theorems 3 and 4 in Lecture II. It depends on the property of continuity possessed by the directions issuing from a point. Let x be an arbi-

trary direction issuing from the vertex of the given angle. Let the angle (x, y) be taken congruent to the angle (a, x) . The angle (a, y) includes the angle (a, c) or is included by it or is congruent to it so that y falls on c . Separate all the directions which issue from the vertex of the angle (a, c) and lie within the angle into two classes of which the first contains all directions such that the double angle (a, y) is included by the given angle (a, c) and the second class contains all the other directions, namely those which are such that the angle (a, y) includes the angle (a, c) or is congruent to it. It remains to show that any direction of the first class precedes any direction of the second class. Let x and x' be two directions such that the angle (a, x) includes the angle (a, x') . Let (x, y) and (x', y') be respectively congruent to (a, x) and (a, x') . Let the angle (y, z) be congruent to (a, x) . Now the angle (a, z) includes the angle (a, y) : for otherwise (x, y) would include (x', z) which is impossible by our hypothesis. Also (x', y') includes (x', z) . Hence y' follows z and *a fortiori* follows y . Therefore if the direction x precedes the direction x' , the direction y must precede y' and conversely. Applying this to the case above, it is seen that every direction in the first class must precede every direction in the second class. Hence there exists some direction, say b , such that no direction in the first class follows it and none in the second class precedes it (by the definition of continuity given in Lecture I). The direction y which corresponds to this direction b cannot either follow c or precede it. Consequently in this case c and y coincide. Hence the angles (a, b) and (b, c) are congruent and the theorem is proved.

Definition. An angle (a, b) is said to be symmetrical to the angle (c, d) when it is congruent to the angle (d, c) .

Theorem 36. If the angle (a, b) is symmetrical to (c, d) , then conversely the angle (c, d) is symmetrical to (a, b) .

Theorem 37. Two angles symmetrical to the same angle or to congruent angles are congruent to one another.

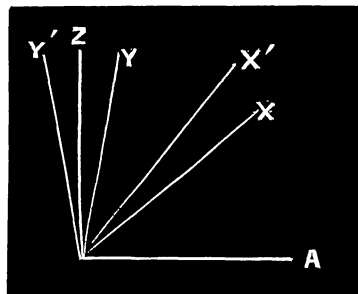
These theorems follow so easily from the definition that no demonstrations will be given. It is important to notice that if two angles are symmetrical we have only to alter the order in which we name the sides of one of them and they become congruent.

Definition of right angle: The congruent angles formed by bisecting a straight angle are called right angles. Or, if two symmetrical adjacent angles are such that the two corresponding sides which are not coincident form the two opposite directions of a line, the angles are said to be right angles.

Theorem 38. From a given point of a given line and on a given side of the line there issues one and only one direction which makes right angles with the given line.

Theorem 39. Any two right angles are either congruent or symmetrical.

Theorem 40. If two lines meet so as to form right angles on either side of either



line, all of the four angles formed are right angles and the lines may be said to be perpendicular.

These are the familiar theorems concerning the erection of perpendiculars. The proofs will be left to the reader.

REPRESENTATION OF REAL AND IMAGINARY LOCI IN THE SAME PLANE.

By PROFESSOR G. W. GREENWOOD. McKendree College.

In elementary analytical geometry, where only the real parts of the locii are represented, it is difficult at times to give satisfactory definitions. Take for example the polar of a point with respect to a conic. If we define it as the locus of the intersection of tangents at points where the conic is cut by any secant through the point, we get when the point is without the conic a portion of the polar upon which no tangents meet; if we define it as a chord of contact, our definition appears strained when the point is within the conic.

I have been using, with some success, a method which I have not seen in any text book, and which is in brief as follows:

Consider the circle $x^2 + y^2 = a^2$. Taking value of x greater than a , numerically, we get $y = \pm ib$, say, where b is real. Plot the points obtained by taking the real factors in the values of y and we get the hyperbola

$$x^2 - y^2 = a^2,$$

which we may draw in red ink. This will contain all imaginary points of the circle of the form (c, ib) .

Now either definition of the polar of a point on $y=0$ will apply; if the point is outside the circle, it is within the hyperbola and its polar is the locus of the intersection of real tangents to the circle or imaginary tangents (*i. e.* those to the hyperbola) at points where the locus is cut by secants through the point. When the point is within the circle, the chord of contact joins the points of tangency of the imaginary tangents.

The radical axis of two circles appears in all cases as the line through two common points, real or imaginary.

Two circles are seen to intersect in two infinite imaginary points, and two concentric circles are said to have double contact at infinity, the tangent there being the circular lines.

The limiting forms of the circle and its hyperbola illustrate graphically the dual interpretation of the equation

$$x^2 + y^2 = 0,$$